

Time-asymmetry of probabilities versus relativistic causal structure: an arrow of time

Bob Coecke and Raymond Lal

University of Oxford, Department of Computer Science, Quantum Group
Wolfson Building, Parks Road, Oxford, OX1 3QD, UK.

There is an incompatibility between the symmetries of causal structure in relativity theory and the signaling abilities of probabilistic devices with inputs and outputs: while time-reversal in relativity will not introduce the ability to signal between spacelike separated regions, this is not the case for probabilistic devices with space-like separated input-output pairs. We explicitly describe a non-signaling device which becomes a perfect signaling device under time-reversal, where time-reversal can be conceptualized as playing backwards a videotape of an agent manipulating the device. This leads to an arrow of time that is identifiable when studying the correlations of events for spacelike separated regions. Somewhat surprisingly, although time-reversal of Popescu-Rörlich boxes also allows agents to signal, it does not yield a perfect signaling device. Finally, we realize time-reversal using post-selection, which could lead experimental implementation.

The bounded speed of light in relativity theory restricts the ability of agents to signal to other agents. For this reason, the study of general probabilistic theories [1, 2] is typically restricted to those theories that do not enable signaling, of which classical and quantum probability are of course the main examples. Due to the symmetries under time-reversal of relativity theory, namely that time-reversal does not introduce the ability to signal between space-like separated regions (see also our discussion below), one may expect that under time-reversal the non-signaling constraint would also be preserved by general probabilistic theories. This turns out not to be the case, not even for classical probability theory. This has important consequences if one were to incorporate probabilistic features within causal structure [3], and provides important new input into the debate on the arrow of time [4]. We discuss this further at the end of this paper.

THE INCONSISTENCY

Given a spacetime manifold, e.g. Minkowski spacetime \mathcal{M} , the spacelike separation of two regions A and B implies that an agent Alice located in A is not able to signal to an agent Bob located in B , and vice versa. We denote this by $A \not\leq B$ and $B \not\leq A$. If A and B are time-like separated then signaling is in principle possible, along the direction of time, but of course not backwards. Hence, in the case that A causally precedes B we have $A \leq B$ and $B \not\leq A$. Now consider the time-reversed spacetime manifold \mathcal{M}^{op} [18], with respect to which we denote the (in)ability of agents to signal by means of \leq^{op} and $\not\leq^{op}$. The symmetry under time-reversal of \mathcal{M} means that $A \leq B$ if and only if $B \leq^{op} A$ (i.e. when A can signal to B in \mathcal{M}), while $A \not\leq B$ if and only if $B \not\leq^{op} A$ (i.e. when A cannot signal to B in \mathcal{M}). In particular, for spacelike separated regions (for which $A \not\leq B$ and $B \not\leq A$), time-reversal does not introduce the ability to signal. This argument straightforwardly extends to more general globally hyperbolic spacetimes [19].

Now we consider a device with two inputs a_I and b_I and two outputs a_O and b_O , Alice and Bob each having access to one input and one output [20]:

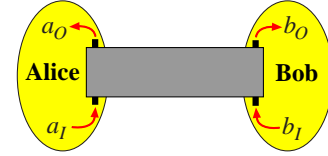


FIG. 1: Two-party probabilistic input-output box.

We assume that this device is non-signaling, and that Alice and Bob remain spatially separated while interacting via the device. We assume that the device can exhibit any non-signaling correlations: classical (including shared randomness), quantum or super-quantum. To such a device we associate a time-reversed device by exchanging the roles of the inputs and the outputs; in the next section we discuss how this may be conceived of operationally. We will show that there exist devices for which time-reversal turns a non-signaling device into a perfect-signaling device.

So, a *non-channel in one time direction* becomes a *perfect channel in the other direction*, contra the time-reversal symmetry of relativity discussed above.

One critical issue here is the precise description of time-reversal for such a probabilistic device. Following standard probability theory one can turn a stochastic matrix with I as givens and O as conclusions into one which has O as givens and I as conclusions via *Bayesian inversion*:

$$P(I|O) = \frac{P(O|I)P(I)}{P(O)}.$$

However this requires knowing the prior probability distribution $P(I)$, which is supposed to be a free choice by the agents. We circumvent this issue: our results do not depend whatsoever on the choice of prior. More specifically, we will show that only the *possibility* of the occurrence of correlations matters, rather than their probability. Then time-reversal has a unique characterization.

‘DETECTING’ THE ARROW OF TIME

We now provide an intuitive operational conception of time-reversal for certain specific situations, which merely consists of rewinding a videotape. Consider the following multi-agents scenario [21]. Each agent has a device with a slot: when one puts a card into the slot, it returns a card. We consider inserting a card as an input process and retrieving a card as an output process. The symmetry of the situation now allows for exchanging the roles of the input process and the output process by time-reversal:

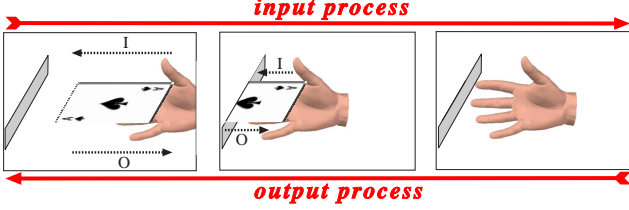


FIG. 2: Exchange of input and output by time-reversal.

While the devices are not connected, they may expose correlations in their behavior which were encoded at the time of their manufacturing.

Assume that one videotapes the input-output process for a number of rounds that is sufficient for statistical purposes. When one plays the tape backwards, the statistics will now be the one obtained via Bayesian inversion. Analysis of the correlations exposes the fact that these devices enable perfect signaling, despite the fact that the agents may be located outside of each other’s light cones.

We can conclude that merely by studying correlations one can ‘detect’ the backward direction of time: this is the direction in which *there exist devices that potentially enable signaling between space-like separated regions*. We discuss this issue in more detail at the end of this paper.

SIGNALING FROM TIME-REVERSAL

Assume that for a two-party I/O-device as described above, a_I , b_I , a_O and b_O take values in $\{0,1\}$. We can assign a 4-by-4 (probabilistic) correlation matrix $g = (g_{a_I, b_I}^{a_O, b_O})$ of which the entries give the probability of obtaining output pair (a_O, b_O) given input pair (a_I, b_I) .

Definition 1. A correlation matrix enables *signaling from Alice to Bob* iff

$$\exists(b_I, b_O) : g_{0, b_I}^{0, b_O} + g_{0, b_I}^{1, b_O} \neq g_{1, b_I}^{0, b_O} + g_{1, b_I}^{1, b_O}. \quad (1)$$

The sums reflect that fact that the value of Alice’s output is not known to Bob, and hence is traced out. So by signaling we mean that, from his input-output pairs, and after a sufficient number of rounds (i.e. in the statistical limit), Bob has obtained information about Alice’s

sequence of inputs. All correlation matrices that we will consider here will be invariant under exchange of the roles of Alice and Bob, hence for simplicity we can restrict ourselves to considering only (non-)signaling from Alice to Bob.

Definition 2. A correlation matrix g is *classical* if there exist 2-by-2 stochastic matrices $\{g_i(a)\}$ and $\{g_i(b)\}$ [22] and $\{P_i\}$ for which $P_i \in [0, 1]$ and $\sum_i P_i = 1$, such that g decomposes as follows [23]:

$$g = \sum_i P_i g_i(a) \times g_i(b) \quad (2)$$

It is well-known that a classical correlation matrix does not enable signaling from Alice to Bob. Now, given g and prior $P(I)$, we rely on Bayesian inversion to construct the time-reverse $g_{P(I)}^T$, explicitly,

$$(g_{P(I)}^T)_{a_O, b_O}^{a_I, b_I} = \frac{g_{a_I, b_I}^{a_O, b_O} \times (P(I))_{a_I, b_I}}{(P(O))^{a_O, b_O}} \quad (3)$$

where

$$(P(O))^{a_O, b_O} = \sum_{a_I, b_I} g_{a_I, b_I}^{a_O, b_O} (P(I))_{a_I, b_I}. \quad (4)$$

The variables a_O and b_O are now treated as the inputs and the variables a_I and b_I are now treated as the outputs. By *perfect signaling* we mean that Bob receives Alice’s input as its output with certainty. We call a prior *total* if it has no zero entries.

Theorem 3. *There exist classical correlation matrices for which the time-reverse for any total prior is signaling. More specifically, each such time reverse of*

$$\tilde{g} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad (5)$$

enables perfect signaling from Alice to Bob, which is achieved when Bob fixes his input to always be 0.

Proof. First we observe that \tilde{g} is indeed classical:

$$\begin{aligned} \tilde{g} = & \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ & + \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

and hence, in particular, \tilde{g} is non-signaling. In order to establish that $\tilde{g}_{P(I)}^T$ enables perfect signaling, note that for any $P(I)$ it will always have the form:

$$\tilde{g}_{P(I)}^T = \begin{pmatrix} a & b & 0 & f \\ 0 & c & e & g \\ 1-a & d & 0 & h \\ 0 & 1-b-c-d & 1-e & 1-f-g-h \end{pmatrix} \quad (6)$$

Assume that Bob fixes his input to be 0. Then when Alice's input is 0 the output will be $(0, 0)$ with probability a and it will be $(1, 0)$ with probability $1 - a$, and when Alice's input is 1 the output will be $(0, 1)$ with probability e and it will be $(1, 1)$ with probability $1 - e$. Hence, Bob's output always perfectly matches Alice's input. \square

THE ROLE OF BAYESIAN INVERSION

The above discussion explicitly involved Bayesian inversion. However, the same conclusion can be obtained merely by looking at *possibilities*, that is, for which pairs of inputs certain outputs are possible. One can represent these possibilities also in a matrix, with 0 standing for impossible and 1 standing for possible. In this case the time-reverse is nothing but the transpose, e.g. for the example of Theorem 3 we have:

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad g^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \quad (7)$$

One can show that the reduced data encoded in these matrices is sufficient to draw the same conclusion as stated in Theorem 3, but now without any reference to priors.

THE CASE OF PR-BOXES

One may (wrongly) have assumed that the ability for time-reverses to be signaling may be a result of non-locality. Since we already established that this phenomenon occurs classically this is not the case. Moreover, for Popescu-Röhrlich (PR) boxes [5], that is, maximally non-local non-signaling correlation matrices, one never achieves perfect signaling under time-reversal.

For a PR-box pr and its time-reverse pr^T we have:

$$pr = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad pr^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad (8)$$

When Bob's output is 0, then Alice's input always matches Bob's. Hence in this case he can deduce Alice's input. However, while we achieve signaling, we don't achieve a perfect channel, since Bob has no control over his output (and neither will Bob instead fixing his input lead to a perfect channel).

REALIZING TIME-REVERSAL

Above we provided an operational conception of time-reversal by means of reversing a videotape. Here we will

show how one can effectively *realize* time-reversal. Evidently, this will require signaling resources, as the outcome may be a signaling device. The signaling resource that we will rely on is *post-selection*, that is, conditioning on an outcome of a probabilistic process.

In [6] it was shown how bipartite states and post-selected bipartite measurements can be used to seemingly reverse the flow of quantum data, and which was later cast as a diagrammatic formalism that encompassed many foundational structures of quantum theory [7, 8]. This has been built on by various authors, e.g. Svetlichny [9], in proposing post-selected quantum teleportation as a means of simulating closed timelike curves (CTCs) [24], which we discuss below.

Now, consider the configuration of Fig. 3 where the lower triangles represent bipartite states and the upper triangles bipartite effects.

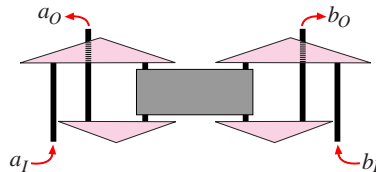


FIG. 3: Realization of the transpose via post-selection.

Following [8], we can make the apparent 'flows' of information explicit by replacing the triangles by wires:

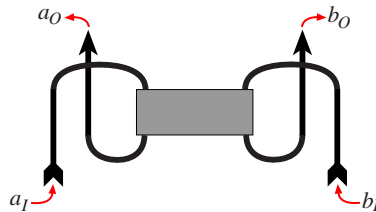


FIG. 4: Information flow of Fig. 3.

Fig. 4 indeed shows that the inputs a_I and b_I are seemingly 'fed' into the outputs of the device.

This leads us to propose a realization of time-reversal as follows. Let the states and effects of Fig 3 be $(1 \ 0 \ 0 \ 1)^T$ and $(1 \ 0 \ 0 \ 1)$ respectively, and consider the original device (depicted by the grey box) to be a probabilistic device as described above. Then it is easily calculated that the entire post-selected device of Fig 3 will produce the transpose of the original correlation matrix. Moreover, Coecke and Spekkens [10] have shown how the configuration of Fig 3 can also realize Bayesian inversion: the states and effects that are now used depend on the prior $P(I)$. This produces a realization of time-reversal that is therefore relative to a particular input-output pair of distributions $(P(I), P(O))$.

Note that if the device of Fig. 3 were to have pairs of qubits as inputs and outputs, and taking the states and effect respectively to be $|00\rangle + |11\rangle$ and $\langle 00| + \langle 11|$, then

we immediately obtain that transpose of the quantum operation. One could rely on the experimental techniques of [11, 12], to effectively realize this in the lab.

Post-selection is hence used both in our proposal for the realization of time-reversal and in the aforementioned proposals for simulation of CTCs [9, 12]. Indeed, the operation described above is actually analogous to the simulation of CTCs by post-selected quantum teleportation, which can be realized by using a state and an effect in the following configuration:

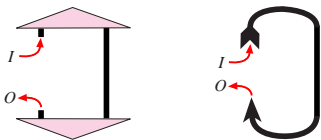


FIG. 5: Realization of a CTC via post-selection.

On the left of Fig. 5, half of a Bell pair $|00\rangle + |11\rangle$ is subject to part of the entangled effect $\langle 00| + \langle 11|$: on the right the wires indeed show that this leads to an apparent flow of information backwards in time through a ‘loop’.

DISCUSSION

We showed that when probabilities are involved, the time-reversed picture (cf. reversing the tape) fundamentally clashes with relativistic light cone structure. Therefore, it seems that one needs to abandon time-symmetry for any theory that combines both relativistic aspects and intrinsic probabilistic aspects. Dually, an analysis of probabilistic correlations enables one to ‘detect’ an arrow of time. Our point is not to argue for the actual existence of a signaling device that can effectively be realized, but rather that asserting ‘too much’ time-related symmetry, as is for example encoded within GR, causes a contradiction with other aspects of physics.

In that respect, it should be noted that in the realization using a video tape and reversing it, we never will observe the perfect signaling device being used to signal: we can only deduce from the statistics that it could be potentially used for that purpose, as Bob’s backward inputs will typically not always be 0. In contrast, in the realization using post-selection we *will* observe signaling for the device of Theorem 3, but note that in general this will depend on the chosen $P(I)$: the $P(O)$ we obtain may not lead to backwards signaling.

For future work, it would be worthwhile to investigate how our work relates to the relationship between information processing and the thermodynamic arrow of time [13], and also its relation to the signaling properties of PR boxes in the presence of CTCs [14]. Also, since causal structure forms the basis for approaches to quantum gravity [15], these approaches may have to be reconsidered in the light of the results in this paper.

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- [18] Note that for general relativistic spacetime \mathcal{G} , the spacetime \mathcal{G}^{op} is also a valid spacetime, since the space of solutions to Einstein’s equations is closed under reversal of time-orientation, i.e. the continuous assignment of ‘future’ and ‘past’ each to one half of the light cone at each point of the manifold; see [16], for further discussion).
- [19] The *causal structure*, i.e. relationships of the kind $A \leq B$ and also $A < B$ between spacetime points, indeed captures much of the essence of relativity: e.g. Malament [17] showed that for globally hyperbolic spacetimes the differentiable structure and the conformal metric can be recovered from the causal structure.
- [20] Devices of this kind are important in quantum foundations and quantum information; e.g. for a Bell-type scenario the inputs constitute the choice of measurements and the outputs the measurement outcomes.
- [21] This particular scenario was suggested to us by Jamie Vicary; it improved on one proposed by us which involved illuminated buttons.
- [22] I.e. $[0, 1]$ -valued entries which in each column sum to 1.
- [23] In the quantum foundations literature this is usually referred to as a *local hidden variable representation*.
- [24] Bennett and Schumacher had earlier suggested, in unpublished work, that post-selected teleportation provides a model of time travel.